An Overview of Relative Trisections

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joint with

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First the closed case ...

Def (Hay-Kirby 12) A $(g,k)$-trisection of a closed smooth 4-manifold $X$ is a decomposition $X = X_1 \cup X_2 \cup X_3$ where

$$X_i \cong \#^k S^1 \times B^3$$

$$\exists X_i \cong \#^2 S^1 \times S^2 = (X_i \cap X_i^+) \cup (X_i \cap X_i^-)$$

A Heegaard splitting of $\exists X_i$:

$$X_i \cap X_i^+ \cong X_i \cap X_i^- \cong H_g \quad \text{genus } g \text{ handlebody}$$

$$X_1 \cap X_2 \cap X_3 \cong S_g \quad \text{genus } g \text{ surface}$$
Def (Hay-Kirby) A $(g,k)$-trisection diagram is a tuple $(S_g, \alpha, \beta, \delta)$ such that

\[
(S_g, \alpha, \beta) \quad \text{and} \quad (S_g, \beta, \delta) \quad \text{are all Heegaard diagrams of}
\]

\# S \times S^2

Thm (Hay-Kirby)

A. Trisection $\mapsto$ \text{Trisection diag} $\xrightarrow{\text{diffco}}$ $\mapsto$ $\text{4-mfds}$ $\xrightarrow{\text{diffco}}$

B. $\text{4-mfds}$ $\xrightarrow{\text{diffco}}$ $\text{Trisection diag}$ $\xrightarrow{\text{stabilization}}$

Examples:

- $S^4$ $(S^2, \phi, \phi, \phi)$
- $S^4$
- $\mathbb{C}P^2$
- $S^2 \times S^2$
Trisections of manifolds with bdry

Def. (A modification of Hay-Kirby) A relative trisection of a 4-mfld $X$ with $\emptyset X \neq \emptyset$

is a decomposition $X = X_1 \cup X_2 \cup X_3$

$s_i \cong \# s^1 \times \mathbb{B}^3$

$s_i \cong \# s^1 \times s^2$

$= (X_i \cap X_{i-1}) \cup (X_i \cap X_{i+1}) \cup (X_i \cap \emptyset X_i)$

Such that ... (Need new terminology)
Def. A sutured mfd is an oriented 3-mfld $Y$ with a decomposition

$$\partial Y = R_- \cup \Gamma \cup R_+$$

(i) Every component of $\Gamma$ is either $T^2$ or $S^1 \times I$

(ii) $\partial R_- \cap \partial R_+ = \emptyset$

Two meaningful examples:

A. $P$ a compact oriented surface with $\partial P = \emptyset$.

$$N = P \times I$$

$$\Gamma = \partial P \times I$$

$$R_+ = P \times \{1\}$$

then $N(P) = (N, \Gamma)$ is a sutured mfd.

B. $P$ as before, $Y$ a 3-mfld $P \subseteq Y$.

$$M = Y \setminus \text{Int}(P \times I)$$

$$\Gamma = \partial P \times I$$

then $Y(P) = (M, \Gamma)$ is a sutured manifold.
**Def (Julhasz)** A sutured Heegaard diagram is a tuple \((\Sigma, \alpha, \beta)\) where - \(\Sigma\) is a surface
- \(\alpha, \beta\) sets of simple closed curves.

**Thm (Julhasz)** Sutured manifolds/diffeo \(\leftrightarrow\) Sutured diagrams/surface diffeo isotopy handle slides

**Pf (sketch)**

\[
M = (\Sigma \times I) \cup \left( \alpha \times \Sigma \times I \right) \cup \left( \beta \times \Sigma \times I \right)
\]

Surgery on \(\Sigma\) along \(\beta\)

Surgery on \(\Sigma\) along \(\alpha\)

The examples from before:

A. \(N(P) = (P \times I, \partial P \times I)\) has diagram \((P, \phi, \phi)\)

B. If \(Y = S^1 \times S^2\) with Heegaard diagram

**diagram with handle slides**

Then \((g, \phi, \phi)\) is a sutured diag for \(Y(P)\).
Back to relative trisections

**Def.** (A modification of a **Stallings-Kirby**) A relative trisection of a 4-manifold $X$ with $\partial X \neq \emptyset$ is a decomposition $X = X_1 \cup X_2 \cup X_3$

$X_i \cong \#^k S^1 \times B^3$

$Y_k = \partial X_i \cong \#^k S^1 \times S^2$

$=(X_i \cap X_i^+) \cup (X_i \cap X_i^-) \cup (X_i \cap \partial X)$

$X_i \cap \partial X \cong N(P)$

$(X_i \cap X_i^+) \cup (X_i \cap X_i^-) \cong \text{a sutured Heegaard splitting of } Y_k(P)$

$X_1 \cap X_2 \cap X_3 = F$ a generic 9-surface with 6 boundary components
The induced structure on $\mathcal{E}X$

In each piece:

$$\mathcal{E}X = \bigcup_{i=1}^{3} x_i \cap \mathcal{E}X \cong \bigcup_{i=1}^{3} I \times P$$

Interaction of the pieces (gluing):

- Hor. bdry: $\{1\} \times P$ in $x_i \cap \mathcal{E}X$ is glued to $\{-1\} \times P$ in $x_i \cap \mathcal{E}X$.
- Vert. bdry: $[0,1] \times P$ in $x_i \cap \mathcal{E}X$ is glued to $[-1,0] \times P$ in $x_i \cap \mathcal{E}X$ via $(t,x) \mapsto (-t,x)$

So:

$$\mathcal{E}X \cong I \times P \setminus (t,x) \sim (-t,x) \text{ for } t, t' \in I \text{ and } x \in \mathcal{E}P$$

an open book decomposition

Thm (Gay - Kirby) Let $X$ be a smooth 4-manifold with non-empty and connected $\mathcal{E}X$. For every OBD of $\mathcal{E}X$, there exists a relative trisection of $X$.

"Open books can be filled with trisections."
A surprising fact about \( \# S^1 \times S^2 \):

Let \( Y_k = \# S^1 \times S^2 \) with:
- Heegaard splitting \( Y_k = H_- \cup H_+ \),
- surface \( S = H_- \cap H_+ \)

Let \( L \subseteq S \) be a collection of curves
s.t \( S \setminus N(L) = \emptyset \cup P \)

\[ g(L) > g(\emptyset) \]

**THM (Castro-Bay\textsuperscript{-}Hay\textsuperscript{-}S.)** "There is a unique way of gluing \( X \times P \) back to \( Y_k(P) \)," in other words:
- \( Y_k(P) \) completely determines \( \# S^1 \times S^2 \)
- A sutured Heegaard diagram for \( Y_k(P) \) completely determines a Heegaard diag. for \( \# S^1 \times S^2 \)

Ex: A genus 5 Heegaard diag. for \( \# S^1 \times S^2 \)

Note: In this case \( Y_k(P) = (S' \times S^2) \# (X \times P) \)
and, in general
\[ Y_k(P) = (\# S' \times S^2) \# (X \times P) \]
for some \( l \).
Def (Castro-Hay-P.) A relative trisection diagram is a tuple $(\Sigma, \alpha, \beta, \delta)$ such that

\[
(\Sigma, \alpha, \beta, \delta) \}
\] are sutured Heegaard diagrams for $Y_k(P)$

(Here $Y_k = \# S^1 \times S^2$)

Examples:

$B^4$  

$x D^2$ has diagram

$B^4$ (Lekili)

$D^2$ bundles over $S^2$
Finding the monodromy of the OB

\[ \alpha - \text{handles} \]

\[ \beta - \text{handles} \]
Thank you!